

ISOGEOMETRIC ANALYSIS FORTHE NUMERICAL SOLUTION OFWAVE EQUATION

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Keywords:Isogemetric Analysis, Cubic B-Spline, Wave Equation, Numerical Method

ABSTRACT

One of the most important issues which is always discussed in science and engineering is solving of differential equations governing the behavior of a system. In recent decades, many numerical methods have been proposed to solve these equations, since only a few of these equations can be directly solved by analytical methods. Among the most common ways, using of the finite element methods in analysis of complex structures and computational mechanics has become commonplace. Following the development of numerical methods, recently a new method of Isogeometric analysis based on NURBS (Non-Uniform Rational B-Spline) functions, provided with the aim of integrating geometry modeling and analysis. The main feature of this method is use of the basic functions of exact geometry modeling as basic functions in analysis of space. According to the importance of this method and also the Isogeometric analysis as a new method known in engineering sciences, in this paper in order to solve the equation of time-dependent we use the third degree of B-Spline (cubic) as the basic functions to numerical solution of differential equation of wave transfer. The results are presented in two space of geometric and time.Finally, some numerical examples are given and the results are compared with exact analytical solution and finite element method results to show the ability and efficiency of this method. The numerical results are found to be in good agreement with the exact solutions. The advantage of the resulting scheme is that the algorithm is very simple so it is very easy to implement. differential equations governing the behavior of a system. In recent decades, many numerical methods have
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analytical met the third degree of B-Spline (cubic) as the basic functions to numerical solution of differential equation of wave transfer. The results are presented in two space of geometric and time.Finally, some numerical examples are *Fh.D. Sudent, Deparment of Civil Enginering, Shahidlahorar University of Kerman, Kerman, Iran*

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INTRODUCTION

The wave equation is an important second-order linear partial differential equation for the description of waves (J. N. Reddy 1991). There are a number of candidate computational geometry technologies that may be used in the discretization methods. This approach is based on Isogeometric analysis method. We may be used in the discretization methods. This approach is based on Isogeometric analysis method. We provide numerical solution to the one-dimensional wave equations, based on IGA method and the cubic B provide numerical solution to the one-dimensional wave equations, based on IGA method and the cubic B-spline interpolation. IGA method used for discretize the space also the B-spline function is applied as an interpolation function in the space dimension. We present a new procedure using periodic cubic B-spline interpolation polynomials to discretize the time derivative. In the proposed approach, a straightforward formulation (Rogers, D.F. 2001) was derived from the approximation of the time derivative of the dependent variable with B-spline basis in a fluent manner.

OVERVIEW OF THE B-SPLINE

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B-spline is a spline function in mathematics and subbranch of numerical analysis that has lowest coverage to degree, leveling and given scope. B-spline functions are simple generalized functions of Bezier curves, so that they do not suffer waste fluctuations with increasing degree. B-spline functions are linear functions of set of basic functions which are made based on De Boor (1978). A B-spline curve can be used to determine degree, control points and knot vector. Coordinate system is ascending in parametric, one- **BEE 7**
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De Boor (1978). A B-spline curve can be used to

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determine degree, control points and knot vector. Coordinate system is ascending in parametric, one-
dimention and that vector. Coordinate system is ascending in parametric, one-

is written in Ω = $\{t_1, t_2, ..., t_{s+d+1}\}$ Where, t_i is 1^{th} node and *i* is index of

olynomial and n is numbers of basic used functions in construct **d** 1, 2, 1, 1, 4 + 1 d is degree of polynomial and n is numbers of passing the action, we can obtain basic spline functions are simple generalized functions of Bezier so that they do not suffer waste fluctuations with in athematics and subbranch of numerical analysis that has lowest
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a spline function in mathematics and subbranch of numerical analysis that has lowest
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treves, so that they do not suffer waste fluctuations with increasing degree. B-spline functions of Exercit

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Now, we can obtain basic spline functions by using De Boor. The zero degree B-spline are defined as follows

$$
B_{i,0}(t) = \begin{cases} 1, & t_i \le t \le t_{i+1} \\ 0, & otherwise \end{cases}
$$
 (1)

and for degree d, it is defined as recursive formula in the following form

spline functions by using De Boor. The zero degree B-spline are defined as
\n
$$
B_{i,0}(t) = \begin{cases} 1, & t_i \le t \le t_{i+1} \\ 0, & otherwise \end{cases}
$$
\n(1)
\nend as recursive formula in the following form
\n
$$
B_{i,d}(t) = \left(\frac{t-t_i}{t_{d+i}-t_i}\right)B_{i,d-1}(t) + \left(\frac{t_{d+i+1}-t}{t_{d+i+1}-t_{i+1}}\right)B_{i+1,d-1}(t)
$$
\n(2)
\npolynomial in $t_i \le t \le t_{i+1}$ interval so that, $B(t)$ and its derivatives from

and open types where each type can have a uniform or non-uniform flavor(Rogers 2001).In a uniform knot vector, knot values are evenly spaced. Here, we preferred to make use of periodic and uniform types. Thus, for a specified order of B-spline, periodic uniform knot vectors yield periodic uniform basis functions for which Where, $B(t)$ is d-degree polynomi
1,2,..., $d-1$ degree are all continuous over the Generally, based on the arrangement
and open types where each type can have
vector, knot values are evenly spaced. Here,
a specified order *^t* . For a uniform knot vector beginning at 0 with integer spacing, usable parameter range is *d n d t t t* and for degree d, it is defined as recursive formula in the following form
 $B_{i,d}(t) = \left(\frac{t-t_i}{t_{d+i} - t_i}\right) B_{i,d-1}(t) + \left(\frac{t_{d+i+1} - t}{t_{d+i+1} - t_{i+1}}\right) B_{i+1,d-1}(t)$ (2)

Where, *B*(*t*) is d-degree polynomial in $t_i \le t \le t_{i+1}$

$$
B_{i,d}(t) = B_{i-1,d}(t-1) = B_{i+1,d}(t+1). \tag{3}
$$

Further, in periodic type each basis function is simply a translation of the other one and the range of nonzero function values spread with increasing order. Thus, the basis function provides support on the

. Thus for the cubic B-spline $(d = 3)$ which we have used in this work, in order to start from $t_0 = 0$, as shown in Fig.1, we have to consider the Basis function from $B_{-3,3}$. interval t_i to t_{i+d+1} .
For a uniform knot vector beginning at 0 with integer spacing, usable parameter range is $t_d \le t \le t_{n-d}$

Interpolation cubic B- Spline function combination of basic three degreesplinefunctions is as follows(Caglar et al. 2006a, b, 2009):

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\n
$$
S_d(t) = \sum_{i=-3}^{n-1} C_i B_{i,3}(t)
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\nB_{i,3}(t) s are cubic (third degree) B-spline
\n1995.
\nC_i (4)
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\nD_{i,3}(t) s are cubic (third degree) B-spline
\n1995.
\nB_{i,4} (5)

 $S_d(t) = \sum_{i=3}^{n-1} C_i B_{i,3}(t)$
Where, C_i s (control points) are unknown coefficients and $B_{i,3}(t)$ *s* are cubic (third degrons (De boor 1978, Yingkang et al. 1995).
We can calculate cubic B- spline function by using re Where, C_i s (control points) are unknown coefficients and $B_{i,3}(t)$ s are cubic (third degree) B- spline functions (De boor 1978, Yingkang et al. 1995).

We can calculate cubic B- spline function by using regression relationship, that is third degree function which is defined as follows:

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$$
S_d(t) = \sum_{i=-3}^{n-1} C_i B_{i,3}(t)
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\n(4)
\ns (control points) are unknown coefficients and $B_{i,3}(t)$ s are cubic (third degree) B- spline
\nor 1978, Yingkang et al. 1995).
\nalculate cubic B- spline function by using regression relationship, that is third degree
\ns defined as follows:
\n
$$
\begin{aligned}\n&(t-t_i)^3, & t_i \le t \le t_{i+1} \\
&B_{i,3}(t) = \frac{1}{6h^3} \begin{cases}\ni^3 + 3h^2(t - t_{i+1}) + 3h(t - t_{i+1})^2 - 3(t - t_{i+1})^3, & t_{i+1} \le t \le t_{i+2} \\
& h^3 + 3h^2(t_{i+3} - t) + 3h(t_{i+3} - t)^2 - 3(t_{i+3} - t)^3, & t_{i+2} \le t \le t_{i+3} \\
& (t_{i+4} - t)^3, & t_{i+3} \le t \le t_{i+4}\n\end{cases}
$$
\n(5)
\n
$$
= T/n \quad \text{or} \quad t_i = t_0 + ih \text{ and } t_i \text{ represents time values which are started from zero } (t_0 = 0) \text{ and h is}
$$
\nThus, we can calculate $B_{0,3}$ from equation (5), simply. So that $B_{i,3}(t) = B_{0,3}(t - ih)$,
\nFigure 1. Show $B_{0,3}$ position among basic functions. Thus, we can calculate the first and
\nes $(B_{0,3}, B_{0,3}^m)$. As mentioned before, we can create $B_{i,3}$ with only one simple transferring

 $S_d(t) = \sum_{i=3}^{n-1} C_i B_i$

Where, C_i s (control points) are unknown coefficions

(De boor 1978, Yingkang et al. 1995).

We can calculate cubic B- spline function by un which is defined as follows:
 $B_{i,3}(t) = \frac{1}{6h^3} \begin{$ **SEE 7**
 $S_d(t) = \sum_{i=3}^{n-1} C_i B_{i,3}(t)$ (4)

bl points) are unknown coefficients and $B_{i,3}(t)$ s are cubic (third degree) B- spline

Yingkang et al. 1995).

cubic B- spline function by using regression relationship, that **SEE 7**

S_a (*t*) = $\sum_{i=3}^{n-1} C_i B_{i,3}(t)$ (4)

Where, C₁ s (control points) are unknown coefficients and $B_{i,3}(t)$ s are cubic (third degree) B- spline

functions (De horo PJPS, Yingkang et al. 1995).

We can calc $B_{i,3}(t) = \frac{1}{6h^3} \begin{cases} (t-t_i)^5, & t_i \le t \le t_{i+1} \\ h^3 + 3h^2(t - t_{i+1}) + 3h(t - t_{i+1})^2 - 3(t - t_{i+1})^3, & t_{i+1} \le t \le t_{i+2} \\ h^3 + 3h^2(t_{i+3} - t) + 3h(t_{i+3} - t)^2 - 3(t_{i+3} - t)^3, & t_{i+2} \le t \le t_{i+3} \end{cases}$ (5)

(*t_{i+4}* - *t*)³, $t_{i+3} \le t \le t_{$ second derivatives ($B'_{0,3}$, $B''_{0,3}$). As mentioned before, we can create $B_{i,3}$ s with only one simple tranfering from $B_{0,3}$. Also, this is true for its derivatives. *t*($\sigma_{i,i} = t$)³, $t_{i+3} \leq t \leq t_{i+4}$

Where, $h = T/n$ *s* $t_i = t_0 + ih$ and *t*, represents time values which are started from zero (interval or *N*. Thus, we can calculate $B_{0,3}$ from equation (5), simply. So that $B_{$ **EXAMPLE 1988**
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 S_{*i*c-3} S _{*ic-3*} $\bigcup_{i=3}^{n-1} C_i B_{i,3}(t)$ (4)

Where, C₁ s (control points) are unknown coefficients and $B_{i-1}(t)$ s are cubic (third degree B - spline

netrions (De boor 1978, Yingkang et al. 1995).
 $\left(\frac{(t_{i+4}-t)}{t} \right)$, $t_{i+3} \le t \le t_{i+4}$
 $h = T/n$ j , $t_i = t_0 + ih$ and t_i represents time values which are started from zero $(t_0 = 0)$ and
 s . Thus, we can calculate $B_{0,3}$ from equation (5), simply. So that $B_{1,3$ and variant coefficients and $R_{i,j}(s)$ are coloric (third degree) B-spline
 $(\tan \theta + \sin \theta)$, then $\theta = \frac{1}{2}$ and $\theta = \frac{$

SOLVING WAVE EQUATION

Most of finite element methods are for time dependent problems based on semi-discretization of problem. Normal differential equation system is resulted of using finite elements in space coordinations. It is used direct integral methods to discretize new equations in time coordination. There are two step to solve these time – dependent equations (discretization method). **the dependent problems based on semi-discretization of**
 the condination is nime coordination. There are two step to solve

method).
 $u_n = S u_{xx}$ (6)
 $u_n = S u_{xx}$ (6)

ace and time descretization. So that, it is done spa sulted of using finite elements in spations in time coordination. Then
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the sum is $u_{tt} = S u_{xx}$
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 \math scale and a testing time cleares in space coordinations. It is
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tization method).

on as follows:
 $u_x = 5 u_x$ (6)

two space and time descretization sons $B_{n,r}$ position among basic functions. Thus, we can calculate the first and
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or its derivatives.
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If it integral

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u_{tt} = S u_{xx} \tag{6}
$$

Wave equations are solved in two space and time descretization. So that, it is done space discretization, firstly and then time discretization. To offer these two discretization steps it was used B spline functions. space and the descretization
ation. To offer these two disc.
SPACE
in one-dimensional is in the form
 $u_{tt} = S^2 u_{xx}$
by an expression of the form
 $\sum_{j}^{r} u_j^e(t) B_j^e(x)$
notion substituting for $u = B_j^e$. *f* (6)
 i terretization. So that, it is done space

wo discretization steps it was used B-

the form (7)

form (8)
 $u = B_j^e$. After writing weak form and
 $\frac{dB_j(x)}{dx} - B_i \left] dx - P_i$ (9) **Propagation of wave equations are solved in two space and time descretization. So that, it is done space
discretization, firstly and then time discretization. To offer these two discretization steps it was used B-
DISCR**

DISCRETIZATION IN GEOMETRIC SPACE SPACE

The propagation of wave with speed β in one-dimensional is in the form

$$
u_{tt} = \mathbf{S}^2 u_{xx} \tag{7}
$$

In this case u_e over an element is interpolated by an expression of the form

retization. To offer these two discretization steps it was used B-
\n**RIC SPACE**
\ned β in one-dimensional is in the form
\n
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u_{tt} = S^2 u_{xx}
$$
\n(7)
\nlated by an expression of the form
\n
$$
u = \sum_{j}^{r} u_j^e(t) B_j^e(x)
$$
\n(8)
\nn function substituting for $u = B_j^e$. After writing weak form and
\nwe:

Where B_j^e are the B-spline interpolation function substituting for $u = B_j^e$. After writing weal regarding to boundary conditions, we have:

and then time discretization. To other these two discretization steps it was used B-
\n**IN GEOMETRIC SPACE**

\nif wave with speed β in one-dimensional is in the form

\n
$$
u_{tt} = S^2 u_{xx}
$$
\n(7)

\nelement is interpolated by an expression of the form

\n
$$
u = \sum_{j}^{r} u_j^e(t) B_j^e(x)
$$
\nand

$$
[M]{\overline{u}} + [K]{u} = {F}
$$
\n
$$
\text{atives}
$$
\n
$$
= \frac{d^2 u_i}{dt^2}
$$
\n
$$
= \frac{
$$

Where in equation (10) \ddot{u}_i are the second derivatives

$$
\ddot{u}_i = \frac{d^2 u_i}{dt^2}
$$

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\n
$$
[M]\{\ddot{u}\} + [K]\{u\} = \{F\}
$$
\n(10)
\nWhere in equation (10) \ddot{u}_i are the second derivatives
\n
$$
\ddot{u}_i = \frac{d^2 u_i}{dt^2}
$$
\nWhere
\n
$$
M_{ij} = \int_{x_i}^{x_{i+1}} B_i^e(x) B_j^e(x), \quad K_{ij} = \int_{x_i}^{x_{i+1}} S^2 \frac{d B_i^e(x)}{dx} \frac{d B_i^e(x)}{dx} dx, \quad F_i^e = \int_{x_i}^{x_{i+1}} B_i^e(x) dx
$$
\n(11)
\nEquation of the discrete space discretization of the wave equation in space is considered to
\ndiscrete in time.
\n**DISCRETIZATION IN TIME SPACE**
\nTo discrete wave equation in time, we consider the equation of discretization in space as follows:
\n
$$
M\ddot{U}_i + KU_i = F_i, \qquad \ddot{u}_k(t) = \sum_{i=3}^{n-1} C_{i,k} B_{i,3}^e(t), \qquad u_k(t) = \sum_{i=3}^{n-1} C_{i,k} B_{i,2}(t). \quad (12)
$$
\nWhere we consider $t_0, t_1, t_2, ..., t_n$ as $n+1$ interval point[0, T]. $k = 1, 2, 3, ..., N$
\n
$$
\forall t_i \in [0, T], \qquad \int_{x_i}^{x_i} C_i \cdot B_i^e(x) dx = \int_{x_i}^{x_i} C_i \cdot B_i^e(x) dx
$$

Equation of the discrete space discretization of the wave equation in space is considered to discrete in time.

DISCRETIZATION IN TIME SPACE

To discrete wave equation in time, we consider the equation of discritization in space as follows:

$$
B_i^e(x)B_j^e(x), \quad K_{ij} = \int_{x_e}^{x_{e+1}} S^2 \frac{dB_i^e(x)}{dx} \frac{dB_j^e(x)}{dx} dx, \quad F_i^e = \int_{x_e}^{x_{e+1}} B_i^e(x) dx
$$
 (11)
\ne discrete space discretization of the wave equation in space is considered to
\n**N IN TIME SPACE**
\ne equation in time, we consider the equation of discretization in space as follows:
\n $M\ddot{U}_t + KU_t = F_t, \qquad \ddot{u}_k(t) = \sum_{i=-3}^{n-1} C_{i,k} B_{i,3}^r(t), \qquad u_k(t) = \sum_{i=-3}^{n-1} C_{i,k} B_{i,2}(t).$ (12)
\n F_1, F_2, \ldots, F_n as $n+1$ interval point [0, T]. $k = 1, 2, 3, \ldots, N$

SET7 [M]
$$
[i\theta]
$$
 + $[K]$ $[u] = [F]$ (10)
\nWhere in equation (10) \ddot{u}_i are the second derivatives
\n
$$
\ddot{u}_j = \frac{d^2 u_j}{\lambda^2}
$$
\nWhere
\n
$$
M_g = \int_{\lambda}^{\lambda} B_i^*(x) B_j^*(x), \quad K_g = \int_{\lambda}^{\lambda} \frac{\sin^2(\lambda)}{dx} \frac{dB_j^*(x)}{dx} dx, \quad F_i^* = \int_{\lambda}^{\lambda} B_i^*(x) dx
$$
 (11)
\nEquation of the discrete space discretization of the wave equation in space is considered to
\ndiscrete in time.
\n**DISCREITZATION IN TIME SPACE**
\nTo discrete wave equation in time, we consider the equation of discretization in space as follows:
\n
$$
M\ddot{U}_j + K U_i = F_j, \qquad \ddot{u}_k(t) = \sum_{i=3}^{n-1} C_{i,k} B_{i,k}^*(t), \qquad u_k(t) = \sum_{i=3}^{n-1} C_{i,k} B_{i,k}(t).
$$
 (12)
\nWhere we consider $t_{n,i_1,i_2,...,i_k}$ as $n+1$ interval point[0,T], $k = 1, 2, 3, ..., N$
\n
$$
\forall t_j \in [0,T],
$$
\n
$$
\begin{vmatrix}\nM_{11} & \cdots & M_{1N} \\
\vdots & \ddots & \vdots \\
M_{N-1} & \cdots & M_{NN}\n\end{vmatrix}\n\begin{vmatrix}\n\sum_{i=1}^{N} C_{i,i} B_{i,i}(t) \\
\vdots & \ddots & \vdots \\
\sum_{i=1}^{N} C_{i,i} B_{i,i}(t) \\
\vdots & \ddots & \vdots \\
\sum_{i=1}^{N} C_{i,i} B_{i,i}(t) \\
\vdots & \ddots & \vdots \\
\sum_{i=1}^{N} C_{i,i} B_{i,i}(t) \\
\vdots & \ddots & \vdots \\
\sum_{i=1}^{N} C_{i,i} B_{i,i}(t) \\
\vdots & \ddots & \vdots \\
\sum_{i=1}^{N} C_{i,i} B_{i,i}(t) \\
\vdots & \ddots & \vdots \\
\sum_{i=1}^{N} C_{i,i} B_{i,i}(t) \\
\
$$

Since in every t_j only three equations of basic equations(B_{j-3} , B_{j-2} , B_{j-1}), their derivatives have values and rest are zero. Thus, we can expand equation (13) in every interval t_i as follow:

$$
\begin{vmatrix}\n1 & \cdots & M_{1N} \\
1 & \cdots & M_{NN}\n\end{vmatrix}\n\begin{vmatrix}\n\sum_{i=3}^{n} C_{i,1} B_{i,3}^{n}(t_{j}) \\
\vdots & \ddots & \vdots \\
\sum_{i=3}^{n} C_{i,N} B_{i,3}^{n}(t_{j})\n\end{vmatrix} + \begin{bmatrix}\nK_{11} & \cdots & K_{1N} \\
\vdots & \ddots & \vdots \\
K_{N1} & \cdots & K_{NN}\n\end{bmatrix}\n\begin{vmatrix}\n\sum_{i=3}^{n} C_{i,1} B_{i,3}(t_{j}) \\
\sum_{i=3}^{n} C_{i,N} B_{i,3}(t_{j})\n\end{vmatrix} = \n\begin{cases}\nF_{1}(t_{j}) \\
F_{N}(t_{j})\n\end{cases}
$$
\n(13)
\n
$$
\begin{aligned}\nT_{11} & \cdots & M_{NN}\n\end{aligned}
$$
\n
$$
\begin{bmatrix}\nT_{21} & \cdots & N_{NN}\n\end{bmatrix}\n\begin{bmatrix}\n\sum_{i=3}^{n} C_{i,1} B_{i,2}(t_{j}) \\
\sum_{i=3}^{n} C_{i,N} B_{i,3}(t_{j})\n\end{bmatrix} = \n\begin{bmatrix}\nF_{1}(t_{j}) \\
F_{N}(t_{j})\n\end{bmatrix}
$$
\n(13)
\n
$$
\begin{aligned}\nT_{11} & \cdots & N_{NN}\n\end{aligned}
$$
\n(13)
\n
$$
\begin{aligned}\nM_{k1} \times (C_{j-3,1} B_{j-3,3}(t_{j}) + C_{j-2,1} B_{j-2,3}(t_{j}) + C_{j-1,1} B_{j-1,3}(t_{j}) + \cdots \\
+ K_{k1} \times (C_{j-3,1} B_{j-3,3}(t_{j}) + C_{j-2,1} B_{j-2,3}(t_{j}) + C_{j-1,1} B_{j-1,3}(t_{j}) + \cdots \\
+ K_{k1} \times (C_{j-3,1} B_{j-3,3}(t_{j}) + C_{j-2,1} B_{j-2,3}(t_{j}) + C_{j-1,1} B_{j-1,3}(t_{j}) + \cdots \\
+ K_{k1} \times (C_{
$$

Table1. values of basic functions and their derivatives in different time

According to the value of the spline function at the knot $(t_i)_{i=0}^n$ which have been determined in T t_i , t_i which have been determined in Table 1 can be summarized as

$$
\left[\Gamma\right]_{N\times N}\left\{C_{j-3}\right\}_{N\times 1} + \left[\mathbf{S}\right]_{N\times N}\left\{C_{j-2}\right\}_{N\times 1} + \left[\mathbf{X}\right]_{N\times N}\left\{C_{j-1}\right\}_{N\times 1} = \left\{F(t_j)\right\}_{N\times 1}
$$
\n(15)

According to the value of the spline function at the knot
$$
(t_i)_{i=0}^n
$$
 which have been determined in Table 1 can be summarized as
\n
$$
[r]_{N \times N} \{C_{j=3}\}_{N \times 4} + [s]_{N \times N} \{C_{j=2}\}_{N \times 4} + [x]_{N \times N} \{C_{j=1}\}_{N \times 4} = \{F(t_j)\}_{N \times 4}
$$
\n(15)
\nwhere α, β and γ are constant values as follows
\n
$$
r_{ij} = \left(\frac{M_{ij}}{\Delta t^2} + \frac{K_{ij}}{6}\right), \quad S_{ij} = \left(\frac{-2M_{ij}}{\Delta t^2} + \frac{2K_{ij}}{3}\right), \quad X_{ij} = \left(\frac{M_{ij}}{\Delta t^2} + \frac{K_{ij}}{6}\right)
$$
\n(16)
\nInitial condition can be written as
\n
$$
\{u(t_0)\} = U_0 \Rightarrow \left\{\sum_{i=3}^{n-1} C_{i,1} B_{i,3}(t_0), \sum_{i=3}^{n-1} C_{i,2} B_{i,3}(t_0), \dots, \sum_{i=3}^{n-1} C_{i,N} B_{i,3}(t_0)\right\}^T
$$

Initial condition can be written as

36.14
\nAccording to the value of the spline function at the knot
$$
(t_1)_{t=0}^n
$$
 which have been determined in Table
\nI can be summarized as
\n
$$
[\Gamma]_{n,x} (C_{j+1})_{x=1} + [s]_{n,x} (C_{j+1})_{n,x} + [u]_{N,x} (C_{j+1})_{n,x} = [F(t_j)]_{x=1}
$$
\n(15)
\nwhere α, β and γ are constant values as follows
\n
$$
\Gamma_q = \left(\frac{M_{q}}{\Lambda^2} + \frac{K_q}{6}\right), \quad S_{ij} = \left(\frac{-2M_{ij}}{\Lambda^2} + \frac{2K_{ij}}{3}\right), \quad X_{ij} = \left(\frac{M_{ij}}{\Lambda^2} + \frac{K_{ij}}{6}\right)
$$
\n(16)
\nInitial condition can be written as
\n
$$
\{u(t_0)\} = U_{ij} = \begin{cases} \sum_{i=1}^{n} C_{i,1} B_{i,2}(t_0), \quad \sum_{i=2}^{n} C_{i,2} B_{i,3}(t_0), \quad \ldots, \sum_{i=3}^{n-1} C_{i,2} B_{i,3}(t_0) \end{cases}
$$
\n(17)
\n
$$
\{u(t_0)\} = V_0 \Rightarrow \left\{\sum_{i=2}^{n-1} C_{i,2} B_{i,3}(t_0), \sum_{i=2}^{n-1} C_{i,2} B_{i,3}(t_0), \ldots, \sum_{i=3}^{n-1} C_{i,2} B_{i,3}(t_0) \right\}^T
$$
\n
$$
\{v_0\} = \frac{-1}{2\Lambda t} \{C_{i}\} + \frac{1}{2\Lambda t} \{C_{i}\}
$$
\n(18)
\nThe spline solution of Eq. (10)
\n
$$
\{v_0\} = \frac{-1}{2\Lambda t} \{C_{i}\} + \frac{1}{2\Lambda t} \{C_{i}\}
$$
\n(19)
\nTherefore
\n
$$
\{F\} = [\mathbb{E} \{C\}
$$
\n(19)
\nWhere
\n
$$
\{F\} = [\mathbb{E} \{C\}
$$
\n(19)
\nWhere
\n
$$
\{F\} = [\mathbb{E} \{C_{i
$$

$$
\{v_0\} = \frac{-1}{2\Delta t} \{C_{-3}\} + \frac{1}{2\Delta t} \{C_{-1}\}\tag{18}
$$

The spline solution of Eq. (10) with the initial condition is obtain by solving the following matrix equation. The matrix is constructed using Eqs(15), (17) and (18). Then as a result, a system of $n + 3$ linear equations in the $n + 3$ unknown $C_{-3}, C_{-2}, \ldots, C_{n-1}$ is obtain. This system can be written in the matrix-vector from as follows. vith the initial condition is obtain by solving t
g Eqs(15), (17) and (18). Then as a result, a sys-
 $c_2,...,C_{n-1}$ is obtain. This system can be writter
 $\{F\} = [\mathbb{E} \mid \{C\}$
 $\{v_0\}, \{F(t_0)\}, \{F(t_1)\}, \{F(t_2)\}, ..., \{F(t_n)\} \}^$ $\Rightarrow \left\{ \sum_{i=3}^{n-1} C_{i,1} B'_{i,3}(t_0), \sum_{i=-3}^{n-1} C_{i,2} B'_{i,3}(t_0), \dots, \sum_{i=-3}^{n-1} C_{i,N} B'_{i,3}(t_0) \right\}^T$
 $\} = \frac{-1}{2\Delta t} \{C_{-3}\} + \frac{1}{2\Delta t} \{C_{-1}\}$

the initial condition is obtain by solving the following

Eqs(15), (17) $\left[\sum_{i=3}^{n} C_{i,1} P_{i,3} S_{0} \right]$, $\left[\sum_{i=3}^{n} C_{i,2} P_{i,3} S_{0} \right]$, \cdots , $\sum_{i=3}^{n} C_{i,N} P_{i,3} S_{0}$
 $= \frac{-1}{2\Delta t} \left\{ C_{-3} \right\} + \frac{1}{2\Delta t} \left\{ C_{-1} \right\}$

the initial condition is obtain by solving the following m

c $\hat{u}(t_0)$ } = $V_0 \Rightarrow \left\{ \sum_{i=3}^{n-1} C_{i,1} B'_{i,3}(t_0) , \sum_{i=3}^{n-1} C_{i,2} B'_{i,3}(t_0) , \dots, \sum_{i=3}^{n-1} C_{i,N} B'_{i,3}(t_0) \right\}^T$
 $\{v_0\} = \frac{-1}{2\Delta t} \{C_{-3}\} + \frac{1}{2\Delta t} \{C_{-1}\}$

4. (10) with the initial condition is obtain by ${v_0} = \frac{-1}{2\Delta t} {C_{-3}} + \frac{1}{2\Delta t} {C_{-1}}$

The spline solution of Eq. (10)

with the initial condition is obtain by solving the fol-

equations in the *n* + 3 unknown $C_{-3}, C_{-2},..., C_{n-1}$ is obtain. This system can be writ (t_0) } = $V_0 \Rightarrow \left\{ \sum_{i=3}^{n-1} C_{i,1} B'_{i,3}(t_0) \right\} \sum_{i=3}^{n-1} C_{i,2} B'_{i,3}(t_0) \dots \sum_{i=3}^{n-1} C_{i,N} B'_{i,3}(t_0) \right\}^T$
 $\{v_0\} = \frac{-1}{2\Delta t} \{C_{-3}\} + \frac{1}{2\Delta t} \{C_{-1}\}$

(10) with the initial condition is obtain by solvin

$$
\{F\} = \left[\mathbb{E}\right] \{C\} \tag{19}
$$

Where

(r

n
$$
C_{-3}, C_{-2},..., C_{n-1}
$$
 is obtain. This system can be written in the matrix-vector
\n
$$
\{F\} = [\mathbb{E}]\{C\}
$$
\n(19)
\n
$$
\{F\} = [\{u_0\}, \{v_0\}, \{F(t_0)\}, \{F(t_1)\}, \{F(t_2)\}, ..., \{F(t_n)\}]^T
$$
\n
$$
\{C\} = [\{C_{-3}\}, \{C_{-2}\}, \{C_{-1}\}, \{C_{0}\}, ..., \{C_{n-1}\}]^T
$$
\n(20)
\nnsion matrix as follows:

$$
\frac{\text{SE E } \times \frac{1}{2}}{\ln 2 \text{ rad}} \text{ as}
$$
\n
$$
\left[\Gamma \Big|_{S \sim R} \left(C_1 \right) \Big|_{S \sim R} \left(C_1 \right) \Big|_{S \sim R} \left[C_1 \right) \Big|_{S \sim R} \left[C_1 \right) \Big|_{S \sim R} \left[C_1 \right] \Big|_{S \sim R} \left[C_1 \right) \Big|_{S \sim R} \left[C_1 \right] \Big|_{S \sim R} \left[\left(\frac{M_g}{\lambda^2} + \frac{K_g}{6} \right) \right] \Big|_{S \sim R} \left[\left(\frac{M_g}{\lambda^2} + \frac{K_g}{6} \right) \right] \Big|_{S \sim R} \left[\left(\frac{M_g}{\lambda^2} + \frac{K_g}{6} \right) \right] \Big|_{S \sim R} \left[\left(\frac{M_g}{\lambda^2} + \frac{K_g}{6} \right) \right] \Big|_{S \sim R} \left[\left(\frac{M_g}{\lambda^2} + \frac{K_g}{6} \right) \right] \Big|_{S \sim R} \left[\left(\frac{M_g}{\lambda^2} + \frac{K_g}{6} \right) \right] \Big|_{S \sim R} \left[\left(\frac{M_g}{\lambda^2} + \frac{K_g}{6} \right) \right] \Big|_{S \sim R} \left[C_1 \right] \Big|_{S \sim R} \left[C_2 \right] \Big|_{S \sim R} \left[C_1 \right] \Big|_{S \sim R} \left[\left(\frac{K_g}{\lambda^2} + \frac{1}{\lambda^2} \right) \left(\frac{K_g}{\lambda^2} \right) \right] \Big|_{S \sim
$$

SEE 7

In fact, which of elements in vectors $\{F\}$, $\{C\}$ is a one-dimension vector and every element of $\mathbb E$ SEE 7
In fact, which of elements in vectors $\{F\}$, $\{C\}$ is a one-dimension vector and every element of α
matrix is $N \times N$ dimension matrix. According to the sparse and bandwidth from the matrix α , in order to
f **SEE 7**

In fact, which of elements in vectors $\{F\}$, $\{C\}$ is a one-dimension vector and every element of α

matrix is $N \times N$ dimension matrix. According to the sparse and bandwidth from the matrix α , in order columns of F matrix and write as follows: ich of elements in vectors $\{F\}$, $\{C\}$ is a ordimension matrix. According to the sparse an
efficients $(C_i s)$, it dose not need to inverse
tirix and write as follows:
 $\begin{bmatrix} \{u_0\} \\ \{v_0\} \end{bmatrix} = \begin{bmatrix} \frac{1}{6} \begin{bmatrix} 1$ ct, which of elements in vectors $\{F\}$, $\{C\}$ is a one-dimension vector and every element of α
 $\alpha \times N$ dimension matrix. According to the sparse and bandwidth from the matrix α , in order to
 α and write as f **ISEE** The fact, which of elements in vectors [*P*], [*C*] is a reading tractor and every denomination \mathbf{F} is a controllable in \mathbf{F} is a controllable controllable controllable and matrix and follows
 Internati which of elements in vectors $\{F\}$, $\{C\}$ is a one-dimension vector and every element of π
 d dimension matrix. According to the space and bandwish from the matrix $\hat{\mathbb{E}}$, and one is a constrained $\mathbb{E}[\mathbb{$

SEE 7
\nIn fact, which of elements in vectors {F}, {C} is a one-dimensional vector and every element of E
\nmatrix is *y*×*N* dimension matrix. According to the sparse and bandwidth from the matrix E, in order to
\nfind unknown coefficients (C, *s*) , it does not need to inverse matrix E. Wecliminate first three rows and
\ncolumns of
$$
\alpha
$$
 matrix and write as follows:
\n
$$
\begin{bmatrix}\n u_0 \\
 v_1 \\
 v_2\n \end{bmatrix}\n= \n\begin{bmatrix}\n u_1 \\
 u_2 \\
 u_1\n \end{bmatrix}\n= \n\begin{bmatrix}\n u_2 \\
 u_1 \\
 u_2\n \end{bmatrix}\n= \n\begin{bmatrix}\n u_1 \\
 u_2 \\
 u_2\n \end{bmatrix}\n= \n\begin{bmatrix}\n u_2 \\
 u_1 \\
 u_2\n \end{bmatrix}\n= \n\begin{bmatrix}\n c_{-3} \\
 c_{-1} \\
 c_{-2}\n \end{bmatrix}\n\end{bmatrix}\n\begin{bmatrix}\n c_{-3} \\
 c_{-1} \\
 c_{-2}\n \end{bmatrix}
$$
\n(22)
\nTo calculate unknown coefficients (see the image) shows the system in block form, we will have:
\n
$$
\begin{bmatrix}\n c_{-1} \\
 c_{-1} \\
 c_{-1}\n \end{bmatrix}\n= \n\begin{bmatrix}\n \left[r - \frac{1}{2} [s] \cdot (r - \frac{1}{2}) [s] \cdot (s) - 2\alpha [s] \cdot (s) \cdot \frac{1}{2} [s] \cdot (s) - \frac{\Delta r}{2} [s] \cdot s \right]\n\end{bmatrix}
$$
\n(23)
\nRegard to $[a_1] = [r] - \frac{1}{2}[s] \cdot [s] \cdot [s]\n\end{bmatrix}$
\nthen to find other vectors of unknown coefficient $\{c_n\}, [c_n] \dots [c_{n-1}]$ we consider $\{r_n\}$ we consider $\{r_n\}$ is given by $[c_n] = [s] \cdot [(r - \frac{1}{2}) [s] \cdot (r - \frac{1}{2}) [s] \cdot (r - \frac{1}{2}) [s] \cdot (r - \frac{1}{2}) \cdot (s - \frac{1}{2}) [s] \cdot (r - \frac{1}{2}) \cdot (s - \frac{1}{2}) [s] \cdot (s - \frac{1}{2}) \cdot (s - \frac{1}{2}) \cdot (s -$

To calculate unknown coefficient vectors , by solving above system in block form, we will have:

$$
\{C_{-3}\} = [\Phi]^{-1} \Big(\{F(t_0)\} - 2\Delta t \Big[x \Big] \{v_0\} - [s] \Big(\frac{3}{2} \{u_0\} - \frac{\Delta t}{2} \{v_0\} \Big) \Big), \tag{23}
$$

Regard to $\left[\Phi\right] = \left(\left[r\right] - \frac{1}{2}\left[s\right] + \left[\chi\right]\right)$

expanding this equation from fourth row to next we will obtain regressive equation to calculate vector s of (23)

(23)

(23)
 $u(x, 0) = \cos(f x), u_t(x, 0) = 0$
 $u(0, t) = \cos(f x), u_t(x, 0) = 0$

(24)
 $u(t) = \cos(f x), u_t(x, 0) = 0$

(25)

(25)

(25)

(25)

(26)

(26)

(26)

(26)

(26)

(26)

(26)

(26)

(27)

(29)

(25)

(26)

(27)

(29)

(29)

(29)

(2 h thrown coefficient $\{C_0\}, \{C_1\}, ..., \{C_{n-1}\}$ we consider $\{F\} = [\mathbb{E}]\{C\}$. By
w to next we will obtain regressive equation to calculate vector s of
-1 we will have :

 $\{C_1\} = \{C_1, C_2, \dots, C_n\}$
 $\{C_2\} = \{S_1\}(C_{n$ coetricient {C₀},{C₁},...,{C_{n+1}} we consider {F_j} = [*u*₁}(C₁}, By
 u t we will obtain regressive equation to calculate vector s of
 u hand, we can determind system displacement and velocity

hand, we can

$$
\{C_i\} = \left[x\right]^{-1} \left(\{F(t_{i+1})\} - \left[r\right]\{C_{i-2}\} - \left[s\right]\{C_{i-1}\}\right) \qquad i = 0, 1, 2, \dots, n-1 \tag{24}
$$

Now, having all unknown coefficient in hand, we can determind system displacement and velocity values in each time.(shojaee 2011) *u*_{*u*} *i* + *u*_{*ux*}, 0 ≤ *x* ≤ 1, *t* ≥ 0 (25)
 *u*_{*u*} *x*, 0 ≤ *x* ≤ 1, *t* ≥ 0 (25)
 u(*x,0*) = *cos(f x), u_t*(*x,0*) = 0 (26)
 u(*0,t*) = *cos(f x),* $u_t(x, 0) = 0$ (26)
 u(*0,t*) = *cos(f t),* \int_0^1 Now, having an unknown coencient in hand, we can determine system displacement and velocity
in each time.(shojaee 2011)
from the conditions
Consider the one-dimensional wave equation in the form
 $u_n = u_{xx}$, $0 \le x \le 1$, $t \$ gressive equation to calculate vec
 $0, 1, 2, ..., n-1$

mind system displacement and v
 $u(x,t)dx = 0$
 $u(x,t)dx = 0$

NUMERICAL EXAMPLES

Consider the one-dimensional wave equation in the form

$$
\mathbf{u}_{tt} = u_{xx}, \quad 0 \le x \le 1, \ t \ge 0 \tag{25}
$$

subject to the initial and boundary conditions

$$
u(x,0) = \cos(f x), \quad u_x(x,0) = 0 \tag{26}
$$

$$
u(0,t) = \cos(f t), \quad \int_0^1 u(x,t)dx = 0 \tag{27}
$$

The exact solution is known as

have equation in the form

\n
$$
u_{tt} = u_{xx}, \quad 0 \le x \le 1, \ t \ge 0
$$
\nlitions

\n
$$
u(x,0) = \cos(f x), \quad u_t(x,0) = 0
$$
\n
$$
u(0,t) = \cos(f t), \quad \int_0^1 u(x,t)dx = 0
$$
\n
$$
u(x,t) = \frac{1}{2}(\cos(f(x+t)) + \cos(f(x-t)))
$$
\n(28)

\nthe step $\Delta t = 0.01$ and the various space steps at the final time $T = 5$ that the solutions become more accurate with the smaller space steps.

are tabulated in Table 2. It can be seen that the solutions become more accurate with the smaller space steps.

X	Exact	FEM	IGA			
	value	$h=0.1$	$h=0.1$	$h=0.02$	$h=0.01$	
0.1	-0.95106	-0.9481	-0.9492	-0.95093	-0.95098	
0.2	-0.80902	-0.8042	-0.8067	-0.80885	-0.80892	
0.3	-0.58779	-0.5861	-0.58621	-0.58767	-0.58772	
0.4	-0.30902	-0.299	-0.30853	-0.30898	-0.309	
0.5		θ	θ	0	Ω	
0.6	0.30902	0.299	0.308526	0.308983	0.308999	
0.7	0.58779	0.5861	0.586209	0.587673	0.587722	
0.8	0.80902	0.8042	0.806698	0.808851	0.808922	
0.9	0.95106	0.9481	0.949196	0.950928	0.950984	

Table 2. Numerical results at the grid points for various mesh sizes

This method can be simply generalized to 2D and 3D wave equations, but as our goal has been just to introduce a new methodology, we have just discussed on 1D wave equation.

CONCLUSIONS

In this study, numerical method was proposed by using cubic B-Spline to solve one-dimensional differential wave equations. In the proposed method used equations of wave differential equations.simple and certain shape of obtained equations shows simple use of these equations. To study obtained results of solving one-dimension wave equations one examples were solved by different time steps and its results were compared with exact solution and results obtained from finite element solution. The results obtained by using the proposed method are similar to exact solution. In compared with finite element method, using cubic B- Spline function led to better and more exact results. Using of B-Spline functions with higher degrees will lead to improve most of characteristics of these functions and increase efficiency of this method. **ENGINE CONSULTER CON** xact solution and results obtained from finite element solution. The results obtained
thod are similar to exact solution. In compared with finite element method, using
led to better and more exact results. Using of B-Splin *Seismology* for

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