ISOGEOMETRIC ANALYSIS FORTHE NUMERICAL SOLUTION OFWAVE EQUATION

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ABSTRACT

One of the most important issues which is always discussed in science and engineering is solving of differential equations governing the behavior of a system. In recent decades, many numerical methods have been proposed to solve these equations, since only a few of these equations can be directly solved by analytical methods. Among the most common ways, using of the finite element methods in analysis of complex structures and computational mechanics has become commonplace. Following the development of numerical methods, recently a new method of Isogeometric analysis based on NURBS (Non-Uniform Rational B-Spline) functions, provided with the aim of integrating geometry modeling and analysis. The main feature of this method is use of the basic functions of exact geometry modeling as basic functions in analysis of space. According to the importance of this method and also the Isogeometric analysis as a new method known in engineering sciences, in this paper in order to solve the equation of time-dependent we use the third degree of B-Spline (cubic) as the basic functions to numerical solution of differential equation of wave transfer. The results are presented in two space of geometric and time.Finally, some numerical examples are given and the results are compared with exact analytical solution and finite element method results to show the ability and efficiency of this method. The numerical results are found to be in good agreement with the exact solutions. The advantage of the resulting scheme is that the algorithm is very simple so it is very easy to implement.

INTRODUCTION

The wave equation is an important second-order linear partial differential equation for the description of waves (J. N. Reddy 1991). There are a number of candidate computational geometry technologies that may be used in the discretization methods. This approach is based on Isogeometric analysis method. We provide numerical solution to the one-dimensional wave equations, based on IGA method and the cubic B-spline interpolation. IGA method used for discretize the space also the B-spline function is applied as an interpolation function in the space dimension. We present a new procedure using periodic cubic B-spline interpolation polynomials to discretize the time derivative. In the proposed approach, a straightforward formulation (Rogers, D.F. 2001) was derived from the approximation of the time derivative of the dependent variable with B-spline basis in a fluent manner.

OVERVIEW OF THE B-SPLINE

B-spline is a spline function in mathematics and subbranch of numerical analysis that has lowest coverage to degree, leveling and given scope. B-spline functions are simple generalized functions of Bezier curves, so that they do not suffer waste fluctuations with increasing degree. B-spline functions are linear functions of set of basic functions which are made based on De Boor (1978). A B-spline curve can be used to determine degree, control points and knot vector. Coordinate system is ascending in parametric, one-dimention and knot vector space which is written in $\Omega = \{t_1, t_2, \ldots, t_{n+d+1}\}$ Where, t_i is Ith node and i is index of node $i = 1, 2, \ldots, n+d+1$ ،d is degree of polynomial and n is numbers of basic used functions in construction of the B-Spline curve.

Now, we can obtain basic spline functions by using De Boor. The zero degree B-spline are defined as follows

$$B_{i,0}(t) = \begin{cases} 1\,, & t_i \le t \le t_{i+1} \\ 0\,, & otherwise \end{cases} \tag{1}$$

and for degree d, it is defined as recursive formula in the following form

$$B_{i,d}(t) = \left(\frac{t - t_i}{t_{d+i} - t_i}\right) B_{i,d-1}(t) + \left(\frac{t_{d+i+1} - t}{t_{d+i+1} - t_{i+1}}\right) B_{i+1,d-1}(t) \tag{2}$$

Where, $B(t)$ is d-degree polynomial in $t_i \le t \le t_{i+1}$ interval so that, $B(t)$ and its derivatives from $1, 2, \ldots, d-1$ degree are all continuous over the entire domain.

Generally, based on the arrangement of the knot vector, B-spline functions are categorized in periodic and open types where each type can have a uniform or non-uniform flavor(Rogers 2001).In a uniform knot vector, knot values are evenly spaced. Here, we preferred to make use of periodic and uniform types. Thus, for a specified order of B-spline, periodic uniform knot vectors yield periodic uniform basis functions for which

$$B_{i,d}(t) = B_{i-1,d}(t-1) = B_{i+1,d}(t+1). \tag{3}$$

Further, in periodic type each basis function is simply a translation of the other one and the range of nonzero function values spread with increasing order. Thus, the basis function provides support on the interval t_i to t_{i+d+1} .

For a uniform knot vector beginning at 0 with integer spacing, usable parameter range is $t_d \le t \le t_{n-d}$. Thus for the cubic B-spline $(d = 3)$ which we have used in this work,in order to start from $t_0 = 0$, as shown in Fig.1, we have to consider the Basis function from $B_{-3,3}$.

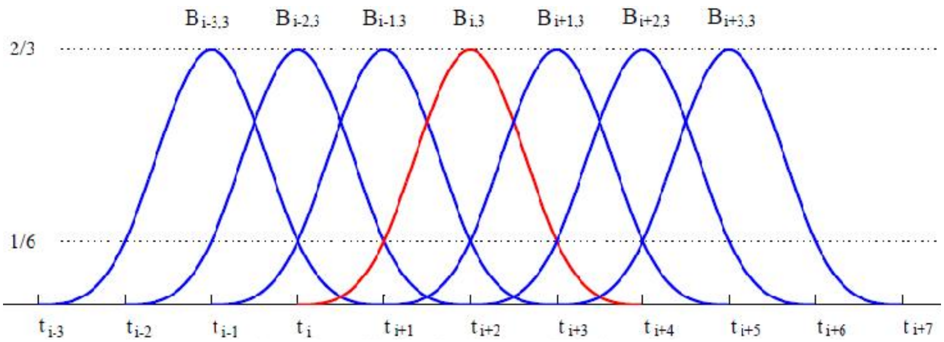

Figure1. cubic periodic B-Spline functions

Interpolation cubic B- Spline function combination of basic three degreesplinefunctions is as follows(Caglar et al. 2006a, b, 2009):

$$S_d(t) = \sum_{i=-3}^{n-1} C_i B_{i,3}(t) \tag{4}$$

Where, C_i s (control points) are unknown coefficients and $B_{i,3}(t)$ s are cubic (third degree) B- spline functions (De boor 1978, Yingkang et al. 1995).

We can calculate cubic B- spline function by using regression relationship, that is third degree function which is defined as follows:

$$B_{i,3}(t) = \frac{1}{6h^3}\begin{cases} (t-t_i)^3, & t_i \le t \le t_{i+1} \\ h^3 + 3h^2(t-t_{i+1}) + 3h(t-t_{i+1})^2 - 3(t-t_{i+1})^3, & t_{i+1} \le t \le t_{i+2} \\ h^3 + 3h^2(t_{i+3}-t) + 3h(t_{i+3}-t)^2 - 3(t_{i+3}-t)^3, & t_{i+2} \le t \le t_{i+3} \\ (t_{i+4}-t)^3, & t_{i+3} \le t \le t_{i+4} \end{cases} \tag{5}$$

Where, $h = T/n$ و $t_i = t_0 + ih$ and t_i represents time values which are started from zero $(t_0 = 0)$ and h is interval or Δt. Thus, we can calculate $B_{0,3}$ from equation (5), simply. So that $B_{i,3}(t) = B_{0,3}(t-ih)$, $i = -3, -2, -1, \ldots$ Figure 1. Shows $B_{0,3}$ position among basic functions. Thus, we can calculate the first and second derivatives ($B'_{0,3}$, $B''_{0,3}$). As mentioned before, we can create $B_{i,3}$ s with only one simple tranfering from $B_{0,3}$. Also, this is true for its derivatives.

SOLVING WAVE EQUATION

Most of finite element methods are for time dependent problems based on semi-discretization of problem. Normal differential equation system is resulted of using finite elements in space coordinations. It is used direct integral methods to discretize new equations in time coordination. There are two step to solve these time – dependent equations (discretization method).
We can consider wave differential equation as follows:

$$u_{tt} = \quad u_{xx} \tag{6}$$

Wave equations are solved in two space and time descretization. So that, it is done space discretization, firstly and then time discretization. To offer these two discretization steps it was used B-spline functions.

DISCRETIZATION IN GEOMETRIC SPACE

The propagation of wave with speed β in one-dimensional is in the form

$$u_{tt} = \quad {}^{2}u_{xx} \tag{7}$$

In this case u_e over an element is interpolated by an expression of the form

$$u = \sum_{j}^{r} u_j^e(t) B_j^e(x) \tag{8}$$

Where B_j^e are the B-spline interpolation function substituting for $u = B_j^e$. After writimg weak form and regarding to boundary conditions, we have:

$$0 = \int_{x_e}^{x_{e+1}} \left[B_i(x) \sum_{j=1}^{r} \frac{d^2 u_j(t)}{dt^2} B_j(x) + \quad \frac{dB_i(x)}{dx} \sum_{j=1}^{r} u_j(t) \frac{dB_j(x)}{dx} - B_i \right] dx - P_i \tag{9}$$

$$[M]\{\ddot{u}\}+[K]\{u\}=\{F\} \tag{10}$$

Where in equation (10) $\ddot{u}_i$ are the second derivatives

$$\ddot{u}_i=\frac{d^2u_i}{dt^2}$$

Where

$$M_{ij}=\int_{x_e}^{x_{e+1}} B_i^e(\mathrm{x})B_j^e(x), \quad K_{ij}=\int_{x_e}^{x_{e+1}} {}^{2}\frac{dB_i^e(x)}{dx}\frac{dB_j^e(x)}{dx}dx, \quad F_i^e=\int_{x_e}^{x_{e+1}} B_i^e(x)dx \tag{11}$$

Equation of the discrete space discretization of the wave equation in space is considered to discrete in time.

DISCRETIZATION IN TIME SPACE

To discrete wave equation in time, we consider the equation of discritization in space as follows:

$$M\ddot{U}_t+KU_t=F_t, \qquad \ddot{u}_k(t)=\sum_{i=-3}^{n-1}C_{i,k}B''_{i,3}(t), \qquad u_k(t)=\sum_{i=-3}^{n-1}C_{i,k}B_{i,2}(t). \tag{12}$$

Where we consider $t_0,t_1,t_2,...,t_n$ as $n+1$ interval point $[0,T]$. $k=1,2,3,\dots,N$

$$\forall t_j\in[0,T],$$

$$\begin{bmatrix} M_{11} & \cdots & M_{1N}\\ \vdots & \ddots & \vdots\\ M_{N1} & \cdots & M_{NN}\end{bmatrix}\begin{Bmatrix}\sum_{i=-3}^{n-1}C_{i,1}B''_{i,3}(t_j)\\ \vdots\\ \sum_{i=-3}^{n-1}C_{i,N}B''_{i,3}(t_j)\end{Bmatrix}+\begin{bmatrix} K_{11} & \cdots & K_{1N}\\ \vdots & \ddots & \vdots\\ K_{N1} & \cdots & K_{NN}\end{bmatrix}\begin{Bmatrix}\sum_{i=-3}^{n-1}C_{i,1}B_{i,3}(t_j)\\ \vdots\\ \sum_{i=-3}^{n-1}C_{i,N}B_{i,3}(t_j)\end{Bmatrix}=\begin{Bmatrix}F_1(t_j)\\ \vdots\\ F_N(t_j)\end{Bmatrix} \tag{13}$$

Since in every t_j only three equations of basic equations(B_{j-3} و B_{j-2} و B_{j-1}), their derivatives have values and rest are zero. Thus, we can expand equation (13) in every interval t_j as follow:

$$\begin{aligned}&M_{k1}\times\left(C_{j-3,1}B''_{j-3,3}(t_j)+C_{j-2,1}B''_{j-2,3}(t_j)+C_{j-1,1}B''_{j-1,3}(t_j)\right)+\cdots\\&+M_{kN}\times\left(C_{j-3,N}B''_{j-3,3}(t_j)+C_{j-2,N}B''_{j-2,3}(t_j)+C_{j-1,N}B''_{j-1,3}(t_j)\right)+\cdots\\&+K_{k1}\times\left(C_{j-3,1}B_{j-3,3}(t_j)+C_{j-2,1}B_{j-2,3}(t_j)+C_{j-1,1}B_{j-1,3}(t_j)\right)+\cdots\\&+K_{kN}\times\left(C_{j-3,N}B_{j-3,3}(t_j)+C_{j-2,N}B_{j-2,3}(t_j)+C_{j-1,N}B_{j-1,3}(t_j)\right)=F_k(t_j)\end{aligned} \tag{14}$$

Table1. values of basic functions and their derivatives in different time

	t_i	t_{i+1}	t_{i+2}	t_{i+3}	t_{i+4}
B_i	0	1/6	2/3	1/6	0
B'_i	0	1/2Δt	0	-1/2Δt	0
B''_i	0	$1/\Delta t^2$	$-2/\Delta t^2$	$1/\Delta t^2$	0

According to the value of the spline function at the knot $(t_i)_{i=0}^{n}$ which have been determined in Table 1 can be summarized as

$$[\sim]_{N\times N}\{C_{j-3}\}_{N\times1}+[\ \ ]_{N\times N}\{C_{j-2}\}_{N\times1}+[\text{¥}]_{N\times N}\{C_{j-1}\}_{N\times1}=\{F(t_j)\}_{N\times1} \tag{15}$$

where α, β and γ are constant values as follows

$$\sim_{ij}=\left(\frac{M_{ij}}{\Delta t^2}+\frac{K_{ij}}{6}\right),\quad {}_{ij}=\left(\frac{-2M_{ij}}{\Delta t^2}+\frac{2K_{ij}}{3}\right),\ \text{¥}_{ij}=\left(\frac{M_{ij}}{\Delta t^2}+\frac{K_{ij}}{6}\right) \tag{16}$$

Initial condition can be written as

$$\begin{aligned}
&\{u(t_0)\}=U_0 \Rightarrow \left\{\sum_{i=-3}^{n-1}C_{i,1}B_{i,3}(t_0)\ ,\ \sum_{i=-3}^{n-1}C_{i,2}B_{i,3}(t_0)\ ,\ \dots\ ,\ \sum_{i=-3}^{n-1}C_{i,N}B_{i,3}(t_0)\right\}^T\\
&\{u_0\}=\frac{1}{6}\{C_{-3}\}+\frac{2}{3}\{C_{-2}\}+\frac{1}{6}\{C_{-1}\}\\
&\{\dot u(t_0)\}=V_0 \Rightarrow \left\{\sum_{i=-3}^{n-1}C_{i,1}B'_{i,3}(t_0)\ ,\ \sum_{i=-3}^{n-1}C_{i,2}B'_{i,3}(t_0)\ ,\ \dots\ ,\ \sum_{i=-3}^{n-1}C_{i,N}B'_{i,3}(t_0)\right\}^T
\end{aligned} \tag{17}$$

$$\{v_0\}=\frac{-1}{2\Delta t}\{C_{-3}\}+\frac{1}{2\Delta t}\{C_{-1}\} \tag{18}$$

The spline solution of Eq. (10)with the initial condition is obtain by solving the following matrix equation. The matrix is constructed using Eqs(15), (17) and (18). Then as a result, a system of $n+3$ linear equations in the $n+3$ unknown $C_{-3},C_{-2},\dots,C_{n-1}$ is obtain. This system can be written in the matrix-vector from as follows.

$$\{F\}=[\cdot\ ]\{C\} \tag{19}$$

Where

$$\begin{aligned}
&\{F\}=\left[\{u_0\},\{v_0\},\{F(t_0)\},\{F(t_1)\},\{F(t_2)\},\dots,\{F(t_n)\}\right]^T\\
&\{C\}=\left[\{C_{-3}\},\{C_{-2}\},\{C_{-1}\},\{C_o\},\dots,\{C_{n-1}\}\right]^T
\end{aligned} \tag{20}$$

And $\cdot$ is a $(n+3)\times(n+3)$ dimension matrix as follows:

$$\begin{bmatrix}
\frac{1}{6}[\mathrm{I}] & \frac{2}{3}[\mathrm{I}] & \frac{1}{6}[\mathrm{I}] & [0] & \cdots & \cdots & \cdots & [0]\\
\frac{-1}{2\Delta t}[\mathrm{I}] & [0] & \frac{1}{2\Delta t}[\mathrm{I}] & [0] & \cdots & \cdots & \cdots & [0]\\
[\sim] & [\ \ ] & [\text{¥}] & [0] & \cdots & \cdots & \cdots & [0]\\
[0] & [\sim] & [\ \ ] & [\text{¥}] & [0] & \cdots & \cdots & [0]\\
\vdots & [0] & [\sim] & [\ \ ] & [\text{¥}] & [0] & & \vdots\\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\
\vdots & \vdots & & [0] & [\sim] & [\ \ ] & [\text{¥}] & [0]\\
[0] & [0] & \cdots & \cdots & [0] & [\sim] & [\ \ ] & [\text{¥}]
\end{bmatrix} \tag{21}$$

In fact, which of elements in vectors $\{F\}$, $\{C\}$ is a one-dimension vector and every element of · matrix is $N \times N$ dimension matrix. According to the sparse and bandwidth from the matrix · , in order to find unknown coefficients $\left(C_i s\right)$, it dose not need to inverse matrix · . Weeliminate first three rows and columns of · matrix and write as follows:

$$\begin{bmatrix} \{u_0\} \\ \{v_0\} \\ \{F(t_0)\} \end{bmatrix} = \begin{bmatrix} \frac{1}{6}[\mathrm{I}] & \frac{2}{3}[\mathrm{I}] & \frac{1}{6}[\mathrm{I}] \\ \frac{-1}{2\Delta t}[\mathrm{I}] & [0] & \frac{1}{2\Delta t}[\mathrm{I}] \\ [\sim] & [\ \ ] & [¥] \end{bmatrix} \times \begin{bmatrix} \{C_{-3}\} \\ \{C_{-2}\} \\ \{C_{-1}\} \end{bmatrix} \tag{22}$$

To calculate unknown coefficient vectors , by solving above system in block form, we will have:

$$\{C_{-3}\} = [\Phi]^{-1}\left(\{F(t_0)\} - 2\Delta t[¥]\{v_0\} - [\ \ ]\left(\frac{3}{2}\{u_0\} - \frac{\Delta t}{2}\{v_0\}\right)\right), \tag{23}$$

Regard to $[\Phi] = \left([\sim] - \frac{1}{2}[\ \ ] + [¥]\right)$

then to find other vectors of unknown coefficient $\{C_0\},\{C_1\},...,\{C_{n-1}\}$ we consider $\{F\} = [\cdot\ ]\{C\}$. By expanding this equation from fourth row to next we will obtain regressive equation to calculate vector s of unknown coefficients $\{C_i\}$, $i = 0,1,...,n-1$ we will have :

$$\{C_i\} = [¥]^{-1}\left(\{F(t_{i+1})\} - [\sim]\{C_{i-2}\} - [\ \ ]\{C_{i-1}\}\right) \qquad i = 0,1,2,\ldots,n-1 \tag{24}$$

Now, having all unknown coefficient in hand, we can determind system displacement and velocity values in each time.(shojaee 2011)

NUMERICAL EXAMPLES

Consider the one-dimensional wave equation in the form

$$\mathrm{u}_{tt} = u_{xx}, \quad 0 \le x \le 1,\ t \ge 0 \tag{25}$$

subject to the initial and boundary conditions

$$u(x,0) = \cos(®\, x), \quad u_t(x,0) = 0 \tag{26}$$

$$u(0,t) = \cos(®\, t), \quad \int_0^1 u(x,t)dx = 0 \tag{27}$$

The exact solution is known as

$$u(x,t) = \frac{1}{2}(\cos(®\,(x+t)) + \cos(®\,(x-t))) \tag{28}$$

Numerical results obtained for time step $\Delta t = 0.01$ and the various space steps at the final time $T = 5$ are tabulated in Table 2. It can be seen that the solutions become more accurate with the smaller space steps.

Table 2. Numerical results at the grid points for various mesh sizes

X	Exact value	FEM	IGA		
		h=0.1	h=0.1	h=0.02	h=0.01
0.1	-0.95106	-0.9481	-0.9492	-0.95093	-0.95098
0.2	-0.80902	-0.8042	-0.8067	-0.80885	-0.80892
0.3	-0.58779	-0.5861	-0.58621	-0.58767	-0.58772
0.4	-0.30902	-0.299	-0.30853	-0.30898	-0.309
0.5	0	0	0	0	0
0.6	0.30902	0.299	0.308526	0.308983	0.308999
0.7	0.58779	0.5861	0.586209	0.587673	0.587722
0.8	0.80902	0.8042	0.806698	0.808851	0.808922
0.9	0.95106	0.9481	0.949196	0.950928	0.950984

This method can be simply generalized to 2D and 3D wave equations, but as our goal has been just to introduce a new methodology, we have just discussed on 1D wave equation.

CONCLUSIONS

In this study, numerical method was proposed by using cubic B-Spline to solve one-dimensional differential wave equations. In the proposed method used equations of wave differential equations.simple and certain shape of obtained equations shows simple use of these equations. To study obtained results of solving one-dimension wave equations one examples were solved by different time steps and its results were compared with exact solution and results obtained from finite element solution. The results obtained by using the proposed method are similar to exact solution. In compared with finite element method, using cubic B-Spline function led to better and more exact results. Using of B-Spline functions with higher degrees will lead to improve most of characteristics of these functions and increase efficiency of this method.

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